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LETTER TO THE EDITOR

Bethe ansatz equations for quantum chains combining different representations of $SU(3)$

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Abstract. The general expression for the local matrix of a quantum chain $L(\theta)$ with the site space in any representation of $SU(3)$ is obtained. This is made by generalizing $L(\theta)$ from the fundamental representation and imposing the fulfilment of the Yang–Baxter equation. With these operators and using a generalization of the nested Bethe ansatz, the Bethe equations for a multistate quantum chain combining two arbitrary representations of $SU(3)$ are obtained.

In the study of integrable quantum systems, chains combining two kinds of spin have aroused great interest lately. The work was pioneered for $SU(2)$ algebra by de Vega and Woynarovich [1]. In this paper a chain-mixing site with spin $\frac{1}{2}$ and 1 and periodic boundary conditions was studied, and the generalization to a chain-combining spin $\frac{1}{2}$ and any other s was suggested. Several subsequent works have been published in which the thermodynamic properties of these systems are studied [2–5].

In this paper, we study an alternating chain, the site states of which are a mixture of any two representations of $SU(3)$. We made an initial approach to this problem in a previous paper [6], where we solved an alternating chain mixture of the two fundamental representations of $SU(3)$ and presented a method, a modification of the nested Bethe ansatz (MNBA), needed to find the Bethe equation (BE) solutions of the problem. The process was as follows. First we sought the general form of the local operator $L(\theta)$ with its auxiliary space in the fundamental representation [7–10] and the site space in any representation of $SU(3)$. This is done by departing from a general form inspired by the local operator $L(\theta)$ with the auxiliary and site space in the fundamental representation of $SU(3)$ and by making that operator the YBE solution. The operator so obtained has several free parameters that are coming from the symmetries of the YBE. With this operator we can form integrable homogeneous chains and find the ansatz equations with usual nested Bethe ansatz (NBA) [11, 12]. Secondly, alternating chains are formed by mixing any two representations of $SU(3)$ and the solutions are formed by applying MNBA [6]. From the results so obtained we can conjecture the BE for chains based on the algebra $SU(n)$.

We denote a representation by the indices of its associated Dynkin diagram (m_1, m_2), where m_1 and m_2 correspond to the $\{3\}$ and $\{\bar{3}\}$ representations respectively. In the figures, a continuous line was used for the fundamental representation $(1, 0)$ and a wavy line for any other representation. Thus, the operators $L(\theta)$ are denoted as indicated in figure 1 and, in order to simplify the writing of the formulae, we will adopt the following identifications: $L(\theta) \equiv L^{(1,0)(1,0)}(\theta)$ and $L'(\theta) \equiv L^{(1,0)(m_1,m_2)}(\theta)$.

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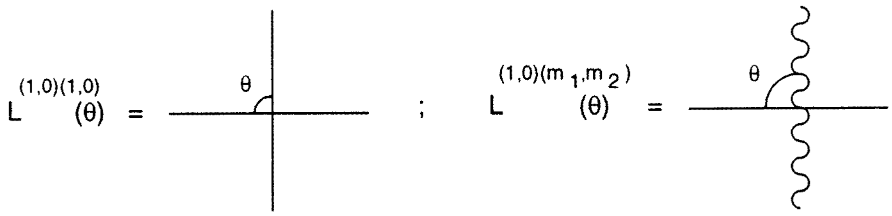

Figure 1.

The operator $L(\theta)$ can be written [6]

$$L(\theta)=\begin{pmatrix} \frac{1}{2}(\lambda^3 q^{-N^\alpha}-\lambda^{-3}q^{N^\alpha}) & \lambda\frac{(q^{-1}-q)}{2}f_1 & \lambda^{-1}\frac{(q^{-1}-q)}{2}[f_2,f_1] \\ \lambda^{-1}\frac{(q^{-1}-q)}{2}e_1 & \frac{1}{2}\left(\lambda^3 q^{-N^\beta}-\lambda^{-3}q^{N^\beta}\right) & \lambda\frac{(q^{-1}-q)}{2}f_2 \\ \lambda\frac{(q^{-1}-q)}{2}[e_2,e_2] & \lambda^{-1}\frac{(q^{-1}-q)}{2}e_2 & \frac{1}{2}(\lambda^3 q^{-N^\gamma}-\lambda^{-3}q^{N^\gamma}) \end{pmatrix} \tag{1}$$

where the parameters λ and q have been taken as the functions of θ and γ

$$\lambda=\mathrm{e}^{\theta/2} \qquad q=\mathrm{e}^{-\gamma} \tag{2}$$

and the N matrices are

$$N^\alpha=\tfrac{2}{3}h_1+\tfrac{1}{3}h_2+\tfrac{1}{3}I \tag{3a}$$
$$N^\beta=-\tfrac{1}{3}h_1+\tfrac{1}{3}h_2+\tfrac{1}{3}I \tag{3b}$$
$$N^\gamma=-\tfrac{1}{3}h_1-\tfrac{2}{3}h_2+\tfrac{1}{3}I \tag{3c}$$

where $\{\epsilon_i f_i q^{\pm h_i}\}$, $i=1,2$ the Cartan generators of the deformed algebra $U_q(SL(3))$.

To obtain the operators $L'(\lambda)$ with the new parameters given in (2), we take (1) as a basis and write

$$L'(\lambda)=\begin{pmatrix} \frac{1}{2}(\lambda^3 q^{-N^\alpha}-\lambda^{-3}q^{N^\alpha}) & \lambda F_1 & \lambda^{-1}F_3 \\ \lambda^{-1}E_1 & \frac{1}{2}(\lambda^3 q^{-N^\beta}-\lambda^{-3}q^{N^\beta}) & \lambda F_2 \\ \lambda E_3 & \lambda^{-1}E_2 & \frac{1}{2}(\lambda^3 q^{-N^\gamma}-\lambda^{-3}q^{N^\gamma}) \end{pmatrix} \tag{4}$$

where the operators $\{E_i, F_i\}$, $i=1,3$ are unknown and will be determined by imposing the YBE

$$R(\lambda/\mu)[L'(\lambda)\otimes L'(\mu)]=[L'(\mu)\otimes L'(\lambda)]R(\lambda/\mu) \tag{5}$$

as shown in figure 2. The $R^{b,d}_{c,a}(\theta)\equiv[L_{a,b}(\theta)]_{c,d}$ is given [10]

$$R(\lambda,\mu)=\begin{pmatrix} a&0&0&0&0&0&0&0&0\\ 0&d&0&b&0&0&0&0&0\\ 0&0&c&0&0&0&b&0&0\\ 0&b&0&c&0&0&0&0&0\\ 0&0&0&0&a&0&0&0&0\\ 0&0&0&0&0&d&0&b&0\\ 0&0&b&0&0&0&d&0&0\\ 0&0&0&0&0&b&0&c&0\\ 0&0&0&0&0&0&0&0&a \end{pmatrix} \tag{6}$$

with

$$a(\lambda,\mu)=\tfrac{1}{2}(\lambda^3\mu^{-3}q^{-1}-\lambda^{-3}\mu^3 q) \tag{7a}$$
$$b(\lambda,\mu)=\tfrac{1}{2}(\lambda^3\mu^{-3}-\lambda^{-3}\mu^3) \tag{7b}$$

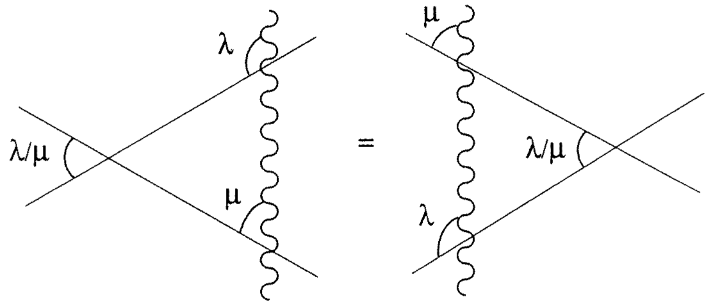

Figure 2.

$$c(\lambda, \mu) = \tfrac{1}{2}(q^{-1} - q)\lambda\mu^{-1} \qquad (7c)$$
$$d(\lambda, \mu) = \tfrac{1}{2}(q^{-1} - q)\lambda^{-1}\mu. \qquad (7d)$$

The relations obtained are

$$E_1 q^{N^\alpha} = q^{-1} q^{N^\alpha} E_1 \qquad (8a)$$
$$E_1 q^{N^\beta} = q q^{N^\beta} E_1 \qquad (8b)$$
$$F_1 q^{N^\alpha} = q q^{N^\alpha} F_1 \qquad (8c)$$
$$F_1 q^{N^\beta} = q^{-1} q^{N^\beta} F_1 \qquad (8d)$$
$$E_2 q^{N^\alpha} = q q^{N^\alpha} E_2 \qquad (8e)$$
$$E_2 q^{N^\beta} = q^{-1} q^{N^\beta} E_2 \qquad (8f)$$
$$F_2 q^{N^\alpha} = q^{-1} q^{N^\alpha} F_2 \qquad (8g)$$
$$F_2 q^{N^\beta} = q q^{N^\beta} F_2 \qquad (8h)$$
$$[E_1, F_1] = (q^{-1} - q)(q^{N^\beta - N^\alpha} - q^{N^\alpha - N^\beta}) \qquad (8i)$$
$$[E_2, F_2] = (q^{-1} - q)(q^{N^\gamma - N^\beta} - q^{N^\beta - N^\gamma}) \qquad (8j)$$
$$E_3 = \frac{1}{(q^{-1} - q)} q^{-N^\beta} [E_1, E_2] \qquad (8k)$$
$$F_3 = \frac{1}{(q^{-1} - q)} q^{N^\beta} [F_2, F_1] \qquad (8l)$$

and besides, the modified Serre relations

$$q^{-1} E_1 E_1 E_2 - (q + q^{-1}) E_1 E_2 E_1 + q E_2 E_1 E_1 = 0 \qquad (9a)$$
$$q E_2 E_2 E_1 - (q + q^{-1}) E_2 E_1 E_2 + q^{-1} E_1 E_2 E_2 = 0 \qquad (9b)$$
$$q^{-1} F_1 F_1 F_2 - (q + q^{-1}) F_1 F_2 F_1 + q F_2 F_1 F_1 = 0 \qquad (9c)$$
$$q F_2 F_2 F_1 - (q + q^{-1}) F_2 F_1 F_2 + q^{-1} F_1 F_2 F_2 = 0 \qquad (9d)$$

should be verified.

It must be noted that the relations (8) are the usual ones for the quantum group $U_q(SL(3))$ while the relations (9) are not the usual ones for the said group and because of this the EYB is not verified if the generators e_i and f_i, pertaining to deformed algebra, are taken as E_i and F_i. This induces us to take

$$F_i = \tfrac{1}{2}(q^{-1} - q) Z_i f_i \qquad (10a)$$
$$E_i = \tfrac{1}{2}(q^{-1} - q) e_i Z_i^{-1} \qquad i = 1, 2 \qquad (10b)$$

where e_i and f_i, $i = 1, 2$ are the generators of $U_q(SL(3))$ in the representation (m_1, m_2) and Z_i are two diagonal operators that were obtained by imposing the verification of the relations (8) and (9). In this way, one obtains the general form of these operators given by

$$Z_1 = q^{a_1 h_1 - \frac{1}{3} h_2 + a_3 I} \tag{11a}$$

$$Z_2 = q^{\frac{1}{3} h_1 + (a_1 + \frac{1}{3}) h_2 + b_3 I} \tag{11b}$$

where the operators h_i, $i = 1, 2$ are the diagonal elements of the algebra $SL(3)$, and a_1, a_3 and b_3 are free parameters that are associated with the transformations that leave the EYB invariant.

The knowledge of the operator L' permits us to find the ansatz of any multistate chain that mixes various representations. For this purpose, the monodromy operator corresponding to the chain to be solved is built; as an example we will use the one which alternates the representations $(1, 0)$ and (m_1, m_2)

$$T^{(alt)}_{a,b}(\theta) = L^{(1)}_{a,a_1}(\theta) L'^{(2)}_{a_1,a_2}(\theta) \ldots L^{(2N-1)}_{a_{2N-2},a_{2N-1}}(\theta) L'^{(2N)}_{a_{2N-1,b}}(\theta) \tag{12}$$

that can be represented graphically as shown in figure 3.

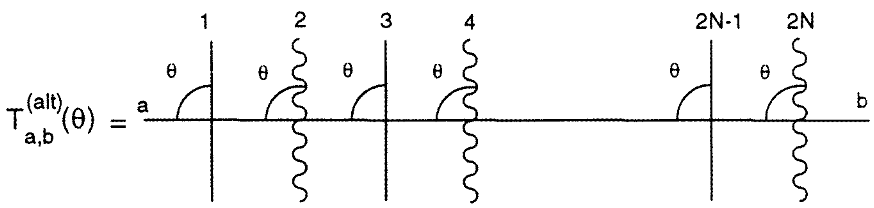

Figure 3.

Using the MNBA [6, 13] the ansatz for the chain can be found. To particularize to each case it is necessary to know the action of the diagonal operators $T^{alt}_{i,i}$ on the vacuum state if the chain is homogeneous or on the vacuum subspace if it is an alternating chain [6]. In both cases, it is always characterized by the highest weight of the representation. Thus, for the representation (m_1, m_2) it will be

$$\Lambda_h = \frac{2m_1 + m_2}{3}\alpha_1 + \frac{m_1 + 2m_2}{3}\alpha_2 \tag{13}$$

where α_1 and α_2 are the simple roots of $SU(3)$.

Through (4), (3a) and (13), together with the commutation rules of $SU(3)$ it was possible to know the action of $L'_{i,i}(\theta)$ on the highest weight, obtaining

$$L'_{1,1}(\theta)\Lambda_h = \sinh(\tfrac{3}{2}\theta + (\tfrac{2}{3}m_1 + \tfrac{1}{3}m_2 + \tfrac{1}{3})\gamma)\Lambda_h \tag{14a}$$

$$L'_{2,2}(\theta)\Lambda_h = \sinh(\tfrac{3}{2}\theta + (-\tfrac{1}{3}m_1 + \tfrac{1}{3}m_2 + \tfrac{1}{3})\gamma)\Lambda_h \tag{14b}$$

$$L'_{3,3}(\theta)\Lambda_h = \sinh(\tfrac{3}{2}\theta + (-\tfrac{1}{3}m_1 - \tfrac{2}{3}m_2 + \tfrac{1}{3})\gamma)\Lambda_h. \tag{14c}$$

It is also applicable for obtaining the action of the operators $L_{i,i}(\theta)$ on the corresponding highest-weight state taking $m_1 = 1$ and $m_2 = 0$. In this way, in the alternate chain that mixes N representations $(1, 0)$ with N representations (m_1, m_2), the BE are given by

$$[g(\mu_k)]^N [\tilde{g}_1(\mu_k)]^N = \prod_{\substack{j=1 \\ j \neq k}}^{r} \frac{g(\mu_k - \mu_j)}{g(\mu_j - \mu_k)} \prod_{i=1}^{s} g(\lambda_i - \mu_k) \tag{15a}$$

$$[\tilde{g}_2(\lambda_k)]^N = \prod_{j=1}^{r} g(\lambda_k - \mu_j) \prod_{\substack{i=1\\ i\neq k}}^{s} \frac{g(\lambda_i - \lambda_k)}{g(\lambda_k - \lambda_i)} \tag{15b}$$

where μ_i, $i = 1, \ldots, r$ and λ_j, $j = 1, \ldots, s$ the roots of the ansatz, the function g is

$$g(\theta) = \frac{\sinh(\frac{3}{2}\theta + \gamma)}{\sinh(\frac{3}{2}\theta)} \tag{16}$$

and $\tilde{g}_1(\theta)$ and $\tilde{g}_2(\theta)$ are obtained from (14) giving

$$\tilde{g}_1(\theta) = \frac{\sinh(\frac{3}{2}\theta + (\frac{2}{3}m_1 + \frac{1}{3}m_2 + \frac{1}{3})\gamma)}{\sinh(\frac{3}{2}\theta + (-\frac{1}{3}m_1 + \frac{1}{3}m_2 + \frac{1}{3})\gamma)} \tag{17a}$$

$$\tilde{g}_2(\theta) = \frac{\sinh(\frac{3}{2}\theta + (-\frac{1}{3}m_1 - \frac{2}{3}m_2 + \frac{1}{3})\gamma)}{\sinh(\frac{3}{2}\theta + (-\frac{1}{3}m_1 + \frac{1}{3}m_2 + \frac{1}{3})\gamma)}. \tag{17b}$$

The procedure can be generalized to chains that mix non-fundamental representations, irrespective of the number of sites and their distribution in the representations. For this purpose, it is necessary to build the monodromy matrix following an analogous process to that used in (12). If we use a broken line for the representation (m'_1, m'_2), the monodromy matrix $T^{(gen)}(\theta)$ can be represented graphically as shown in figure 4.

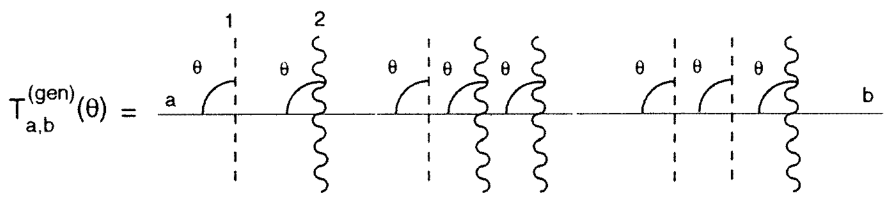

Figure 4.

The eigenvalues for the local operators on the highest-weight states, in straightforward notation are

$$\bar{l}_{1,1}(\theta) = \sinh(\tfrac{3}{2}\theta + (\tfrac{2}{3}m_1 + \tfrac{1}{3}m_2 + \tfrac{1}{3})\gamma) \tag{18a}$$

$$\bar{l}_{2,2}(\theta) = \sinh(\tfrac{3}{2}\theta + (-\tfrac{1}{3}m_1 + \tfrac{1}{3}m_2 + \tfrac{1}{3})\gamma) \tag{18b}$$

$$\bar{l}_{3,3}(\theta) = \sinh(\tfrac{3}{2}\theta + (-\tfrac{1}{3}m_1 - \tfrac{2}{3}m_2 + \tfrac{1}{3})\gamma) \tag{18c}$$

$$\tilde{l}_{1,1}(\theta) = \sinh(\tfrac{3}{2}\theta + (\tfrac{2}{3}m'_1 + \tfrac{1}{3}m'_2 + \tfrac{1}{3})\gamma) \tag{18d}$$

$$\tilde{l}_{2,2}(\theta) = \sinh(\tfrac{3}{2}\theta + (-\tfrac{1}{3}m'_1 + \tfrac{1}{3}m'_2 + \tfrac{1}{3})\gamma) \tag{18e}$$

$$\tilde{l}_{3,3}(\theta) = \sinh(\tfrac{3}{2}\theta + (-\tfrac{1}{3}m'_1 - \tfrac{2}{3}m'_2 + \tfrac{1}{3})\gamma). \tag{18f}$$

By calling the number of sites in the representations (m_1, m_2) and $(m'_1, m'_2) N_1$ and N_2 respectively, we found the eigenvalue of the transfer matrix for this general chain

$$\begin{aligned}\Delta(\theta) = {} & [\bar{l}_{1,1}(\theta)]^{N_1}[\tilde{l}_{1,1}(\theta)]^{N_2} \prod_{j=1}^{r} g(\mu_j - \theta) \\ & + \prod_{j=1}^{r} g(\theta - \mu_j) \Bigg[[\bar{l}_{2,2}(\theta)]^{N_1}[\tilde{l}_{2,2}(\theta)]^{N_2} \prod_{i=1}^{s} g(\lambda_i - \theta) \\ & + [\bar{l}_{3,3}(\theta)]^{N_1}[\tilde{l}_{3,3}(\theta)]^{N_2} \prod_{l=1}^{r} \frac{1}{g(\theta - \mu_l)} \prod_{i=1}^{s} g(\theta - \lambda_i) \Bigg]\end{aligned} \tag{19}$$

and the BE are

$$[\bar{g}_1(\mu_k)]^{N_1}[\tilde{g}_1(\mu_k)]^{N_2} = \prod_{\substack{j=1\\ j\neq k}}^{r} \frac{g(\mu_k-\mu_j)}{g(\mu_j-\mu_k)} \prod_{i=1}^{s} g(\lambda_i-\mu_k) \tag{20a}$$

$$[\bar{g}_2(\lambda_k)]^{N_1}[\tilde{g}_2(\lambda_k)]^{N_2} = \prod_{j=1}^{r} g(\lambda_k-\mu_j) \prod_{\substack{i=1\\ i\neq k}}^{s} \frac{g(\lambda_i-\lambda_k)}{g(\lambda_k-\lambda_i)} \tag{20b}$$

where $\tilde{g}_1$ and $\tilde{g}_2$ are given in (15*a*,*b*) and $\bar{g}_1$ and $\bar{g}_2$ are the same as the previous ones but (m_1, m_2) is replaced by (m'_1, m'_2).

In the light of this, the generalization for the case of mixed chains with more than two different representations seems simple, although the physical models that they represent will be less local and the interaction more complex.

In a non-homogeneous chain combining different representations of $SU(n)$, each representation introduces $(n-1)$ functions (that we call source functions). Each solution will have $(n-1)$ sets of equations (with the same number of dots in its Dynkin diagram). The first member of the equations will be a product of the respective source functions powered to the number of sites of each representation and the second a product of source functions similar to (20).

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